33rd Annual DOD Cost Analysis Symposium
Advanced Training Session

General-Error Regression for Deriving Cost-Estimating Relationships

Stephen A. Book and Philip H. Young
The Aerospace Corporation
P.O. Box 92957
Los Angeles, CA 90009-2957
(310) 336-8655
stephen.a.book@aero.org

Abstract

In most cost models, cost-estimating relationships (CERs) are derived by explicitly computing the classical least-squares linear regression equation $Y = a + bX + E$, where Y represents the cost, X the numerical value of a cost driver, E an error term whose variance does not depend on the numerical value of X, and a and b numerical coefficients derived from historical cost and technical data. Coefficients in nonlinear forms such as $Y = aX^bE$ are traditionally derived by first taking logarithms of both sides of the equation, thereby reducing the original nonlinear expression to the classical linear form $\log(Y) = \log(a) + b\log(X) + \log(E)$. This standard approach to nonlinear regression analysis suffers from a number of well-documented weaknesses in addition to the fact that the error of estimation is expressed in meaningless units ("log dollars"). A second weakness is that the analyst is forced to assume an additive-error (uniform dollar value across the board) model whenever historical data indicate a linear relationship between cost driver and cost, but a multiplicative-error (a percentage of the estimate) model whenever a nonlinear relationship is indicated. A further weakness *a priori* excludes from consideration certain potentially attractive nonlinear forms, such as $Y = a+bX^c$, because a logarithmic (or any other reasonable) transformation fails to reduce the problem to the classical linear-regression format. All known weaknesses of the traditional approach can be circumvented by applying "general-error" regression, which allows the analyst to determine the optimal coefficients for <u>any</u> curve shape and to choose the error model <u>independently</u> of the CER's shape. The optimal (error-minimizing) solution is found by sequential computer search rather than by explicit solution of simultaneous equations analogous to the classical "normal" equations.

CERs that comprise the latest version (Version 7, August 1994) of the U.S. Air Force's *Unmanned Space Vehicle Cost Model* ("USCM-7") have been statistically derived from historical cost data using general-error least-squares regression. In the case of USCM-7 CER development, the multiplicative-error model is used for <u>all</u> CERs, whether their shape be linear or one of several nonlinear types. The optimality criterion for CER selection is minimization of percentage standard error of the estimate.

Biographies of Authors

Dr. Stephen A. Book is Distinguished Engineer at The Aerospace Corporation, El Segundo, CA, serving as the Corporation's principal technical authority on costs of space and space-related systems. During 1989-1995 he held the position of Director, Resource and Requirements Analysis Department, leading the Corporation's efforts in cost research, estimating, and analysis. Dr. Book has given numerous presentations on cost-risk analysis and other statistical aspects of cost and economics to DoD and NASA Cost Symposia, the CAIG/IDA Cost Research Symposium held annually at the Institute for Defense Analyses, the AF/NASA/ESA Space Systems Cost Analysis Group (SSCAG), and professional societies such as SCEA, ISPA, MORS, and AIAA. In prior positions at The Aerospace Corporation, he worked on statistical test design, analysis of test data, and system optimization for a wide variety of Air Force space programs. More recently, he has served as an independent reviewer of NASA programs, for example as a member of the 1998 Cost Assessment and Validation Task Force on the International Space Station ("Chabrow Committee") and the 1998-99 National Research Council Committee on Space Shuttle Upgrades. Dr. Book earned his Ph.D. in mathematics, with concentration in probability and statistics, at the University of Oregon.

Mr. Philip H. Young retired in 1993 after 32 years at The Aerospace Corporation, El Segundo, CA, doing mathematical analysis and systems optimization for all aspects of Air Force space programs. Serving in the Resource Analysis Department from 1990 until his retirement, he made several innovative contributions to Air Force methodology in cost-risk analysis and other statistical aspects of cost analysis. Mr. Young earned his B.S. and M.S. degrees in mathematics at California State University, Los Angeles. He is now Director of Research for Lori Associates of Costa Mesa, CA, an independent consulting firm specializing in the statistical analysis of cost and risk issues.

General-Error Regression for Deriving Cost-Estimating Relationships

Stephen A. Book and Philip H. Young
The Aerospace Corporation
P.O. Box 92957
Los Angeles, CA 90009-2957
(310) 336-8655
stephen.a.book@.aero.org

Advanced Training Session
33rd Annual DoD Cost Analysis Symposium
Williamsburg, VA
1-4 February 2000

THE AEROSPACE CORPORATION

Contents

- The Statistical Approach to CER Development
- How it Was Done in the Past and Why
- Limitations of the Traditional Methods
- General-Error Regression
- Constrained Optimization Solutions
- Examples
- Summary

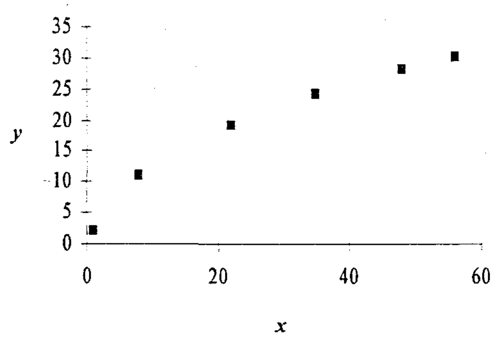
DERIVING PPT # 2

THE AEROSPACE CORPORATION

Historical Data

COST	TECHNICAL PARAMETER
y_1	x_1
y_2	x_2
y_3	x_3
.	.
.	.
.	.
y_n	x_n

SCATTERGRAM



THE AEROSPACE CORPORATION

Mathematical Formulation

- y = Cost
  x = Technical Parameter
- Factor CER: $y = ax$
- Linear CER: $y = a + bx$
- "Nonlinear" CERs: $y = ax^b$
  $y = ab^x$
  $y = a + bx^c$
- a, b, c are Constant Coefficients Derived From Historical Data

THE AEROSPACE CORPORATION

Contents

Statistical CER Development

- Gather, Standardize ("Normalize") Historical Cost Data
  - System, subsystem, component costs
  - System, subsystem, component technical parameters
  - Programmatic parameters
- Use Statistical Methods, Primarily Regression, to Find a Mathematical Relationship Expressing Costs in Terms of Technical Parameters
  - Derive cost-estimating relationship (CER) that minimizes "error of estimation
  - Cost estimation using CERs often called "Parametric Estimation", with some common technical parameters being weight, power, memory, downlink capacity, etc.

THE AEROSPACE CORPORATION

Traditional Solution in the Linear Case

- Linear "Additive-Error" Model

$$y = a + bx + \varepsilon$$

i.e., True Cost = Estimated Cost + Error of Estimation

- Ordinary Least-Squares (OLS) Regression Minimizes Sum of Squared Errors
  - Actual cost for data point i is y_i
  - Estimated cost for data point i is $a + bx_i$
  - Error of estimation for data point i is $\varepsilon_i = y_i - (a + bx_i)$
  - Values of a and b minimize $\sum (y_i - a - bx_i)^2 = \sum \varepsilon_i^2$
- OLS Solution

$$b = \frac{n\sum x_i y_i - (\sum x_i)\ (\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2}$$

$$a = \frac{\sum y_i - b\sum x_i}{n}$$

DERIVING PPT # 7

THE AEROSPACE CORPORATION

Error of Estimation in the Linear Case

- Predict Cost Using Formula $y = a + bx$
- Error Made in Estimating Actual Cost y_i as $a + bx_i$ (corresponding to technical parameter x_i) Equals

$$\varepsilon_i = y_i - a - bx_i$$

- Sum of Squared Errors $\sum (y_i - a - bx_i)^2 = \sum \varepsilon_i^2$ is as Small as Possible if a and b are Chosen as Above
- Sample Bias = $\sum (y_i - a - bx_i) = \sum \varepsilon_i = 0$ Exactly, and it Turns Out that Resulting Estimates are Unbiased
- Standard Error of the Estimate is

$$\text{SEE} = \sqrt{\frac{1}{n-2}\sum (y_i - a - bx_i)^2} = \sqrt{\frac{1}{n-2}\sum \varepsilon_i^2}$$

DERIVING PPT # 8

THE AEROSPACE CORPORATION

Traditional Solution in the Nonlinear Case

- **Consider the Nonlinear Form: $y = ax^b$**
- **Take Logarithms of Both Sides: $\log y = \log a + b \log x$**
- **Logarithmic Transformation of Nonlinear Form Into Linear Form Permits Use of OLS Mathematics to Solve Nonlinear Problem**
- **Determine a and b to Predict $\log y$**
  - Use OLS setup: $\log y = \log a + b \log x + E$
  - $E = \log y - (\log a + b \log x)$ is error of estimation in predicting logarithm of cost
  - Choose values for a and b that minimize the sum of squared errors $\sum(\log y_i - \log a - b \log x_i)^2 = \sum E_i^2$

DERIVING PPT # 9

THE AEROSPACE CORPORATION

Nonlinear OLS-Based Solution

- **Predict $\log y = \log a + b \log x = A + b \log x$ where**

$$b = \frac{n \sum(\log x_i)(\log y_i) - (\sum \log x_i)(\sum \log y_i)}{n \sum(\log x_i)^2 - (\sum \log x_i)^2}$$

$$A = \log a = \frac{\sum \log y_i - b \sum \log x_i}{n}$$

$$(a = 10^{\log a} = 10^A)$$

- $\sum (\log y_i - \log a - b \log x_i)^2 = \sum E_i^2 = \sum (\log \varepsilon_i)^2$ **is as small as possible if a and b are chosen as above**
- **Standard Error of Estimate is reported as**

$$\text{SEE} = \sqrt{\frac{1}{n-2}\sum(\log y_i - \log a - b \log x_i)^2} = \sqrt{\frac{1}{n-2}\sum E_i^2}$$

DERIVING PPT # 10

THE AEROSPACE CORPORATION

Error of Estimation in the Nonlinear Case

- In Order for Logarithms to Work, Nonlinear CER Model Must be "Multiplicative-Error" Model, i.e.,

  $$y = ax^b \varepsilon$$

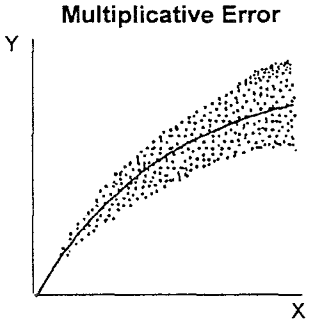
  i.e., True Cost = Estimated Cost *x* Error of Estimation
- Then Applying Logarithms Yields Additive-Error Model for Predicting Logarithm of Cost

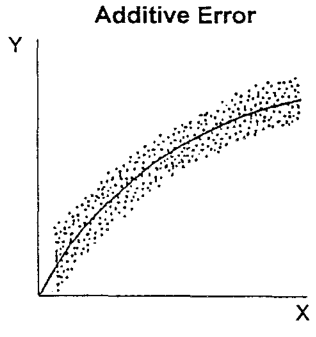
  $$\log y = \log a + b \log x + \log \varepsilon$$
- Recall that What is Minimized Here is $\sum (\log \varepsilon_i)^2$ NOT $\sum \varepsilon_i^2$, as in the Linear Case

DERIVING PPT # 11

THE AEROSPACE CORPORATION

Alternative Error Specifications





Reference: H.L. Eskew and K.S. Lawler, "Correct and Incorrect Error Specifications in Statistical Cost Models," Journal of Cost Analysis, Spring 1994, page 107.

DERIVING PPT # 12

THE AEROSPACE CORPORATION

What's Wrong With This Procedure?

- Bad: Minimizing $\sum (\log \varepsilon_i)^2$ is Not the Same as Minimizing $\sum \varepsilon_i^2$
  - a and b values turn out to be different
  - Error of estimating logarithm of cost is minimized
  - Error is not expressed in meaningful units ("log dollars")
- Worse: Standard Error in Nonlinear Case $\sqrt{\frac{1}{n-2}\sum(\log \varepsilon_i)^2}$ Cannot be Compared with Standard Error in Linear Case $\sqrt{\frac{1}{n-2}\sum \varepsilon_i^2}$ to See which Functional Form is the Better Estimator
- Worst: If You Choose Nonlinear Functional Form, You Must Assume Multiplicative-Error Model; if You Choose Linear Functional Form, You Must Assume Additive-Error Model
- Worstissimo: You do not Have Access to the Functional Form $y = a + bx^c$
- Furthermore: The Estimates are Biased Low

DERIVING PPT # 13

THE AEROSPACE CORPORATION

Theory vs. Practice

" . . . given the nature of our intellectual commerce . . ., to lack a persuasive theory is to lack something crucial – the means by which our experience of individual works [e.g., CERs] is joined to our understanding of the values they signify."

- Art critic Hilton Kramer, The New York Times, (Sunday, April 28, 1974, Arts & Leisure, Section 2, page 19)

" . . . frankly, these days, without a theory to go with it, I can't *see* a painting [or *use* a CER]."

- Social commentator Tom Wolfe, The Painted Word, New York: Farrar, Straus & Giroux, ©1975, page 4.

DERIVING PPT # 14

THE AEROSPACE CORPORATION

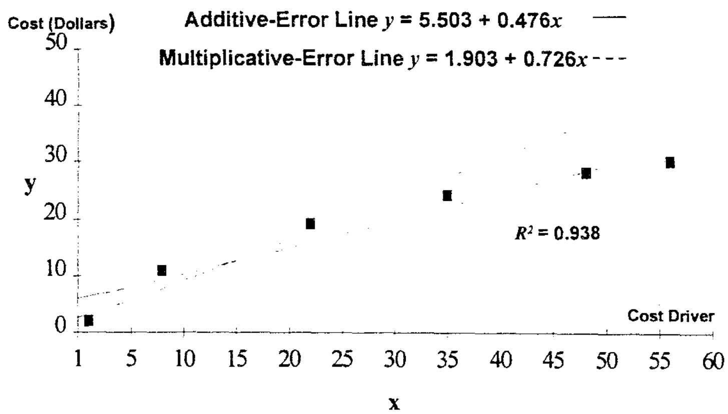
General-Error Regression to the Rescue!

- **General-Error Regression Does Not Use Logarithms**
  - Predict cost, not logarithm of cost
  - Compare and rank all functional forms in magnitude by standard errors
  - Choose appropriate error model (additive or multiplicative) independently of functional form
- **General-Error Regression Takes Advantage of Modern Computing Capability**
  - Nonlinear OLS method is part of the historical residue of the pre-computer age
  - Least-squares minimization problem need not be solved explicitly to get formulas for a and b (as in the linear additive-error case)
  - Sequential-search techniques based on Newton's or simplex method find error-minimizing values of a and b
  - All functional forms can be considered, even $y = a + bx^c$

Straight-Line Fits Compared

Sample Data

x	y
1	2
8	11
22	19
35	24
48	28
56	30



Additive-Error Model

- Provides CERs with Errors Expressed as a Constant Number of Dollars, namely
  Actual Cost = Estimate ± Error in Dollars,
- Formally, Actual Cost Equals Estimate plus the Error, i.e.,
  $$y = f(x) + \varepsilon$$
- Therefore Error = Difference between Actual and Estimate
  $\varepsilon = y - f(x)$ = Estimate - Actual
- Minimum Percentage Error (MPE) CERs: Choose $f(x)$'s Coefficients so that Sum of Squared Percentage Errors
  $$\sum \varepsilon_i^2 = \sum [y_i - f(x_i)]^2$$
  is as Small as Possible

DERIVING PPT # 17

THE AEROSPACE CORPORATION

Multiplicative-Error Model

- Most Appropriate Model for Cost Estimating Usage because it Provides CERs for which
  Actual Cost = Estimate ± Percentage of Estimate
- Formally, Actual Cost Equals Estimate times Error, i.e.,
  $$y = f(x) \times \varepsilon$$
- Therefore Error = Ratio of Actual to Estimate
  $$\varepsilon = \frac{y}{f(x)} = \frac{\text{Actual}}{\text{Estimate}}$$
- Minimum Percentage Error (MPE) CERs: Choose $f(x)$'s Coefficients so that Sum of Squared Percentage Errors
  $$\sum (\varepsilon_i - 1)^2 = \sum \left[\frac{y_i - f(x_i)}{f(x_i)} \right]^2$$
  is as Small as Possible

DERIVING PPT # 18

THE AEROSPACE CORPORATION

Details of MPE Procedure

- Using $f(x) = a + bx^c$ for Purposes of Illustration . . .
- Multiplicative-Error Model
  - $y = \left(a + bx^c\right) \times \varepsilon$
  - $\varepsilon = \dfrac{y}{a + bx^c}$
  - For best results, ε should be as close to one as possible
  - Choose a, b, c so that $\sum(\varepsilon_i - 1)^2 = \sum\left(\dfrac{y_i - a - bx_i^c}{a + bx_i^c}\right)^2$ is as small as possible
  - Apply computation-intensive optimization techniques of numerical analysis, e.g., Newton's or simplex method
- MPE-Capable Software
  - SYSTAT
  - SAS
  - Tablecurve
  - S-Plus (for Sun Workstation)
  - Excel Solver
  - Others . . .

DERIVING PPT # 19

THE AEROSPACE CORPORATION

What About Bias?

- Sample Percentage Bias of MPE CERs = $\frac{1}{n}\sum\left[\dfrac{f(x_i) - y_i}{f(x_i)}\right]$
- Estimates Appear to be Biased High
  - Is this bad?
  - The reason may be that $\sum\left(\dfrac{y_i - f(x_i)}{f(x_i)}\right)^2$ will be smaller if the $f(x)$ values are larger
- Applying "Iteratively Reweighted Least Squares" (IRLS) Adjusts MPE CERs to Eliminate the Bias
  - Sample percentage bias goes to zero
  - But sample standard error increases about 10 - 20% in cases we tested
  - Why? Because you cannot serve two masters
  - This is called the Minimum Unbiased Percentage Error (MUPE) procedure
- IRLS-Derived CERs are Referred to Minimum Unbiased Percentage Error (MUPE) CERs

DERIVING PPT # 20

THE AEROSPACE CORPORATION

Details of MUPE (IRLS) Procedure

- Using $f(x) = a + bx^c$ for Purposes of Illustration . . .
- Generate Sequences of Coefficients
  - $a_1, a_2, a_3, \ldots$
  - $b_1, b_2, b_3, \ldots$
  - $c_1, c_2, c_3, \ldots$
  - Given a_j, b_j, c_j calculate $a = a_{j+1}, b = b_{j+1}, c = c_{j+1}$ by minimizing

$$\sum \left[\frac{y_i - a - bx_i^{\,c}}{a_j + b_j x_i^{\,c_j}} \right]^2$$

- a, b, c = Respective Limits of Sequences if Sequences Converge
  - MUPE CER is $y = a + bx^c$
  - Not the same coefficient values for a, b, and c as in MPE CER

DERIVING PPT # 21 THE AEROSPACE CORPORATION

Why MUPE (IRLS) CERs Are Unbiased

- Recall that IRLS Sequentially Minimizes

$$\sum \left[\frac{y_i - a - bx_i^{\,c}}{a_j + b_j x_i^{\,c_j}} \right]^2$$

by Varying a, b, and c, while a_j, b_j, and c_j Remain Fixed

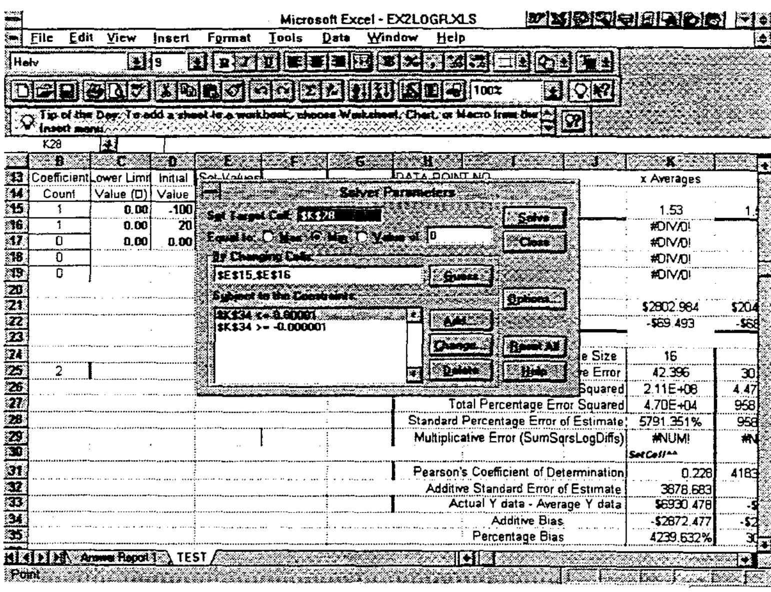
- Careful Analysis of Exactly How IRLS Works Reveals that it is a Weighted Additive-Error Minimization Process at No Point of Which is Percentage Error Explicitly Minimized
- This Brings Up the Question of Whether we Can Achieve the Same Effect (i.e, Unbiasedness) at Lower Cost in Percentage Error by Applying a Method that Explicitly Minimizes Percentage Error

DERIVING PPT # 22 THE AEROSPACE CORPORATION

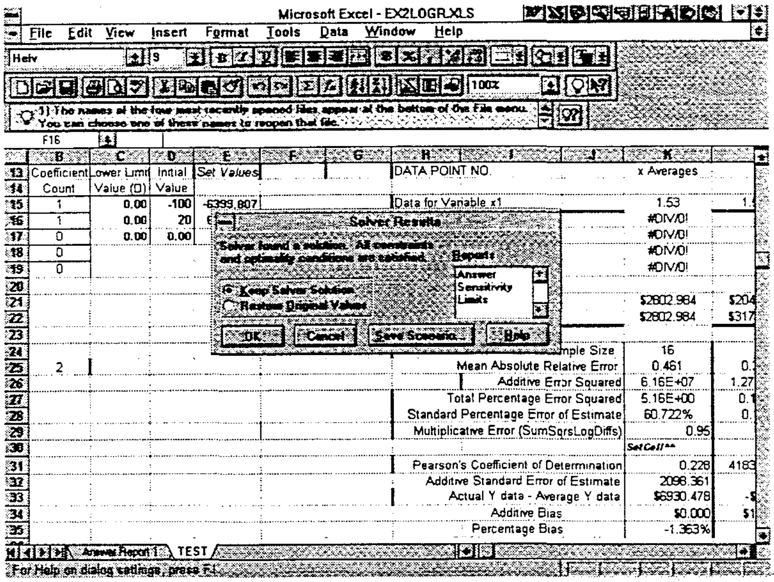
The ZPB/MPE Procedure

- **Minimize Percentage Error, Subject to Constraint that Sample Percentage Bias be Zero**
- **Mathematicians Call This "Constrained Optimization"**
  - Constrain sample percentage bias to zero
  - Then Minimize sample percentage error
- **ZPB/MPE Leads to CER with Zero Sample Percentage Bias ("ZPB") and the Smallest Possible Sample Percentage Error ("MPE") that Can be Achieved with Zero Bias**
- **It Logically Follows that the Sample Percentage Error for a ZPB/MPE-Derived CER Cannot Be Larger Than That of the Corresponding MUPE-Derived CER**
  - Because MUPE(IRLS) does not explicitly optimize anything ...
  - But ZPB/MPE CER is optimal with respect to MPE criterion, given ZPB

Excel Spreadsheet Before Optimization



Excel Spreadsheet After Optimization



THE AEROSPACE CORPORATION

Percentage Standard Error Comparison
(Sample Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	50.550%	56.979%	56.979%
$y = a + bx$	31.687%	32.814%	32.804%
$y = a + b \log x$	15.521%	15.665%	15.647%
$y = bc^x$	42.048%	48.830%	44.769%
$y = bx^c$	13.807%	13.924%	13.896%
$y = a + bx^c$	0.199%	0.199%	0.199%

THE AEROSPACE CORPORATION

Percentage Bias Comparison
(Sample Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	21.294%	0.000%	0.000%
$y = a + bx$	6.694%	0.000%	0.000%
$y = a + b \log x$	1.606%	0.000%	0.000%
$y = bc^x$	11.788%	0.000%	0.000%
$y = bx^c$	1.271%	0.000%	0.000%
$y = a + bx^c$	0.000%	0.000%	0.000%

THE AEROSPACE CORPORATION

Bias vs. Estimating Error in Three Examples

- Example 1 Data

y	x
357.79	6.90
823.70	11.79
652.31	10.23
278.81	6.74
1066.73	16.70
437.44	8.05
1219.83	23.46
368.38	16.50

Example 2 Data

y	x
2045.42	38.59
1619.62	28.92
2079.58	23.30
918.85	21.11
1231.13	17.54
3641.96	27.60
1314.85	16.20
1128.39	34.89
3989.48	46.61
3130.08	65.90
376.47	14.63
9028.31	50.10
2786.09	38.10
2497.71	73.21
2051.06	64.81
7008.74	41.60

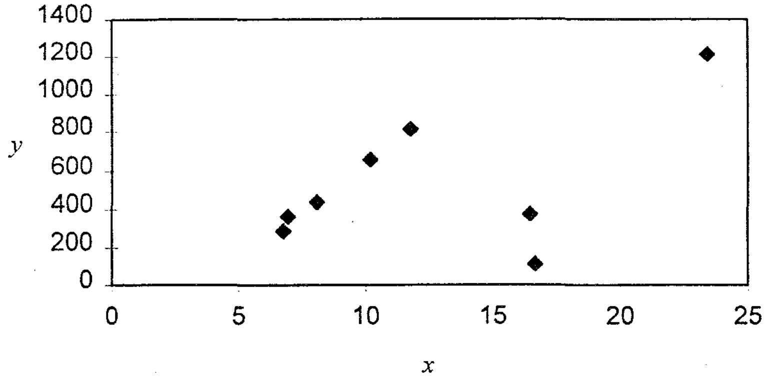
Example 3 Data

y	x
134.96	4.18
2.05	0.32
5.35	0.57
64.64	2.34
32.85	0.50
95.42	2.70
66.22	4.54
112.23	4.42
29.24	0.55
123.09	0.79
28.66	0.20
16.93	0.80
218.20	2.40

- Several multiplicative-error CERs $y = f(x)$ calculated for each data set using all four methods
- Sample bias and standard error compared on later charts

THE AEROSPACE CORPORATION

Graph of Example 1 Data



10000

250

DERIVING PPT # 29

THE AEROSPACE CORPORATION

Percentage Standard Error Comparison
(Example 1 Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	27.814%	28.806%	28.806%
$y = a + bx$	29.539%	31.040%	30.555%
$y = a + b \log x$	26.884%	28.268%	27.644%
$y = bc^x$	34.135%	35.793%	35.732%
$y = bx^c$	29.932%	31.270%	30.992%
$y = a + bx^c$	29.839%	30.438%	29.994%

DERIVING PPT # 30

THE AEROSPACE CORPORATION

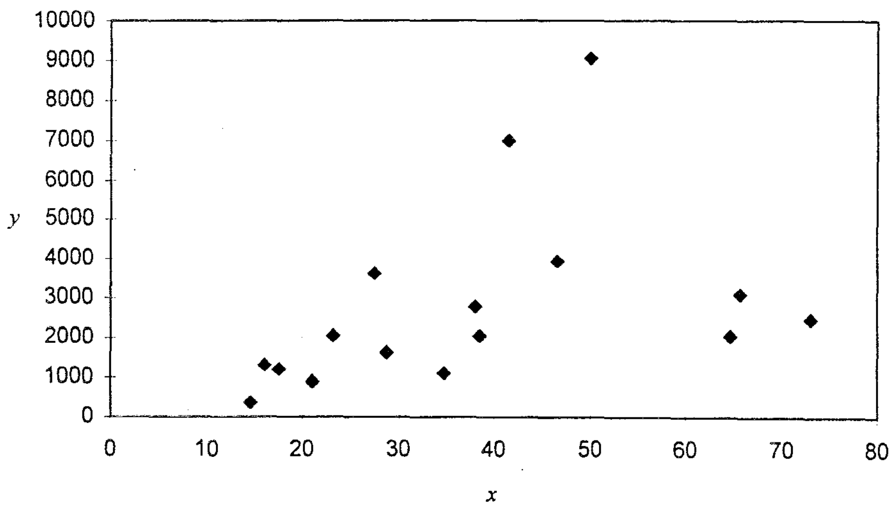
Percentage Bias Comparison
(Example 1 Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	6.770%	0.000%	0.000%
$y = a + bx$	6.551%	0.000%	0.000%
$y = a + b \log x$	5.415%	0.000%	0.000%
$y = bc^x$	8.739%	0.000%	0.000%
$y = bx^c$	6.720%	0.000%	0.000%
$y = a + bx^c$	5.501%	0.000%	0.000%

DERIVING PPT # 31

THE AEROSPACE CORPORATION

Graph of Example 2 Data



DERIVING PPT # 32

THE AEROSPACE CORPORATION

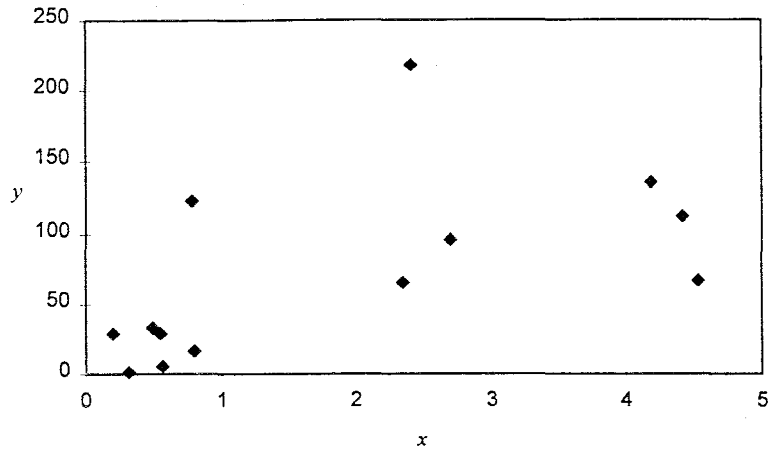
Percentage Standard Error Comparison
(Example 2 Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	53.851%	63.109%	63.109%
$y = a + bx$	52.258%	61.390%	59.902%
$y = a + b \log x$	53.275%	60.978%	58.181%
$y = bc^x$	56.791%	71.589%	67.032%
$y = bx^c$	53.321%	64.011%	61.519%
$y = a + bx^c$	53.125%	61.270%	60.461%

Percentage Bias Comparison
(Example 2 Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	27.197%	0.000%	0.000%
$y = a + bx$	23.892%	0.000%	0.000%
$y = a + b \log x$	19.080%	0.000%	0.000%
$y = bc^x$	28.222%	0.001%	0.000%
$y = bx^c$	24.876%	0.000%	0.000%
$y = a + bx^c$	22.864%	0.010%	0.000%

Graph of Example 3 Data



THE AEROSPACE CORPORATION

DERIVING PPT # 35

Percentage Standard Error Comparison (Example 3 Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	69.711%	93.878%	93.878%
$y = a + bx$	68.176%	88.529%	87.527%
$y = a + b \log x$	63.393%	78.072%	78.034%
$y = bc^x$	69.578%	95.767%	90.554%
$y = bx^c$	65.260%	83.208%	81.599%
$y = a + bx^c$	66.426%	87.059%	82.786%

THE AEROSPACE CORPORATION

DERIVING PPT # 36

Percentage Bias Comparison
(Example 3 Data)

FUNCTION	MPE	MUPE (IRLS)	ZPB/MPE
$y = bx$	44.866%	0.000%	0.000%
$y = a + bx$	39.347%	0.000%	0.000%
$y = a + b \log x$	34.018%	0.000%	0.000%
$y = bc^x$	40.963%	0.000%	0.000%
$y = bx^c$	36.037%	0.000%	0.000%
$y = a + bx^c$	33.940%	0.000%	0.000%

Summary

- General-Error Regression Offers Important Advantages Over Traditional Method of CER Development
  - Permits user to design CERs of any functional form
  - Allows user to select error model independently of functional form
  - Achieves true minimization of error of estimation
  - Constrained optimization enhancements offer user sample unbiasedness options in trade for increase in sample estimating error
- User-Friendly General-Error-Capable Software Available
  - Excel Solver
  - Others
- Do Not Use Old Methods Unless all you have is a Slide Rule

Backup Charts

DERIVING PPT # 39

THE AEROSPACE CORPORATION

MPE Regression Functions Fit To Sample Data

(Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	1.280	–	0.938	50.550%	25.964	21.294%	17.259
$y = a + bx$	1.901	0.726	–	0.938	31.687%	8.016	6.694%	3.463
$y = a + b \log x$	1.974	14.083	–	0.953	15.521%	2.931	1.606%	-0.082
$y = bc^x$	–	9.393	1.024	0.843	42.048%	5.140	11.788%	1.517
$y = bx^c$	–	2.331	0.661	0.986	13.807%	2.229	1.271%	0.555
$y = a + bx^c$	-4.735	6.735	0.408	1.000	0.199%	0.053	0.000%	0.000

DERIVING PPT # 40

THE AEROSPACE CORPORATION

MUPE (IRLS) Regression Functions Fit To Sample Data

(Minimum Unbiased Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	1.007	–	0.938	56.979%	15.859	0.000%	9.538
$y = a + bx$	1.705	0.684	–	0.938	32.814%	6.433	0.000%	2.087
$y = a + b \log x$	1.967	13.799	–	0.953	15.665%	3.075	0.000%	-0.430
$y = bc^x$	–	6.097	1.035	0.794	48.830%	7.224	0.000%	0.867
$y = bx^c$	–	2.264	0.667	0.985	13.924%	2.172	0.000%	0.392
$y = a + bx^c$	-4.735	6.735	0.408	1.000	0.199%	0.053	0.000%	0.000

DERIVING PPT # 41

THE AEROSPACE CORPORATION

ZPB/MPE Regression Functions Fit To Sample Data

(Zero Percentage Bias, Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	1.007	–	0.938	56.979%	15.859	0.000%	9.538
$y = a + bx$	1.774	0.677	–	0.938	32.804%	6.239	0.000%	1.960
$y = a + b \log x$	1.942	13.857	–	0.953	15.647%	3.045	0.000%	-0.386
$y = bc^x$	–	8.287	1.024	0.843	44.769%	4.989	0.000%	-0.902
$y = bx^c$	–	2.302	0.661	0.986	13.896%	2.036	0.000%	0.306
$y = a + bx^c$	-4.735	6.735	0.408	1.000	0.199%	0.053	0.000%	0.000

DERIVING PPT # 42

THE AEROSPACE CORPORATION

MPE Regression Functions Fit To Example 1 Data

(Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	56.230	–	0.615	27.814%	233.629	6.770%	54.855
$y = a + bx$	-90.531	64.916	–	0.615	29.539%	276.396	6.551%	73.294
$y = a + b \log x$	-1092.557	1698.923	–	0.613	26.884%	254.320	5.415%	54.540
$y = bc^x$	–	260.120	1.077	0.600	34.135%	559.982	8.739%	74.393
$y = bx^c$	–	48.286	1.063	0.614	29.932%	529.305	6.720%	63.426
$y = a + bx^c$	-2687.409	1978.300	0.220	0.616	29.839%	542.688	5.501%	56.573

DERIVING PPT # 43

THE AEROSPACE CORPORATION

MUPE (IRLS) Regression Functions Fit To Example 1 Data

(Minimum Unbiased Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	52.423	–	0.615	28.806%	221.955	0.000%	7.094
$y = a + bx$	-5.840	52.981	–	0.615	31.040%	240.324	0.000%	8.244
$y = a + b \log x$	-860.3832	1429.466	–	0.613	28.268%	237.706	0.000%	1.586
$y = bc^x$	–	249.431	1.073	0.601	35.793%	244.058	0.000%	4.475
$y = bx^c$	–	56.644	0.968	0.615	31.270%	238.050	0.000%	3.229
$y = a + bx^c$	1559.539	-4651.290	-0.687	0.601	30.438%	264.495	0.000%	0.385

DERIVING PPT # 44

THE AEROSPACE CORPORATION

ZPB/MPE Regression Functions Fit To Example 1 Data

(Zero Percentage Bias, Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	52.423	–	0.615	28.806%	221.955	0.000%	7.093
$y = a + bx$	-83.805	60.584	–	0.615	30.555%	254.116	0.000%	25.676
$y = a + b \log x$	-1033.185	1606.707	–	0.613	27.644%	242.640	0.000%	16.333
$y = bc^x$	–	237.804	1.077	0.600	35.732%	510.465	0.000%	10.787
$y = bx^c$	–	45.043	1.063	0.614	30.992%	496.906	0.000%	15.444
$y = a + bx^c$	1683.517	-5028.200	-0.671	0.601	29.994%	476.266	0.000%	12.127

DERIVING PPT # 45

THE AEROSPACE CORPORATION

MPE Regression Functions Fit To Example 2 Data

(Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	103.347	–	0.155	53.851%	2593.275	27.197%	1092.629
$y = a + bx$	-1407.114	149.416	–	0.155	52.258%	3280.473	23.892%	1422.044
$y = a + b \log x$	-7509.386	7196.110	–	0.228	53.275%	2260.779	19.080%	664.103
$y = bc^x$	–	646.949	1.048	0.033	56.791%	6529.836	28.222%	2761.985
$y = bx^c$	–	17.981	1.493	0.117	53.321%	3695.230	24.876%	1581.652
$y = a + bx^c$	-17450.901	10513.852	0.201	0.215	53.125%	2797.620	22.864%	1151.282

DERIVING PPT # 46

THE AEROSPACE CORPORATION

MUPE (IRLS) Regression Functions Fit To Example 2 Data

(Minimum Unbiased Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	75.240	–	0.155	63.109%	2162.711	0.000%	33.128
$y = a + bx$	-573.736	94.755	–	0.155	61.390%	2359.934	0.000%	195.002
$y = a + b \log x$	-6012.004	5777.361	–	0.228	60.978%	2092.200	0.000%	-2.581
$y = bc^x$	–	984.738	1.026	0.076	71.589%	2475.958	0.001%	123.747
$y = bx^c$	–	52.855	1.100	0.147	64.011%	2302.691	0.000%	98.948
$y = a + bx^c$	-249490.469	242709.865	0.011	0.228	61.270%	2195.171	0.010%	85.026

DERIVING PPT # 47

THE AEROSPACE CORPORATION

ZPB/MPE Regression Functions Fit To Example 2 Data

(Zero Percentage Bias, Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	75.240	–	0.155	63.109%	2162.711	0.000%	33.131
$y = a + bx$	-1070.728	113.710	–	0.155	59.902%	2547.036	0.000%	412.513
$y = a + b \log x$	-7668.655	6976.566	–	0.228	58.181%	2142.595	0.000%	169.958
$y = bc^x$	–	464.720	1.048	0.033	67.032%	4558.375	0.000%	1189.594
$y = bx^c$	–	13.533	1.492	0.117	61.519%	2798.987	0.000%	490.235
$y = a + bx^c$	-16890.187	11118.874	0.165	0.218	60.461%	2287.873	0.000%	246.214

DERIVING PPT # 48

THE AEROSPACE CORPORATION

MPE Regression Functions Fit To Example 3 Data

(Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	95.024	–	0.317	69.711%	172.632	44.866%	106.169
$y = a + bx$	41.531	38.942	–	0.317	68.176%	77.789	39.347%	42.827
$y = a + b \log x$	95.056	97.325	–	0.420	63.393%	60.236	34.018%	30.311
$y = bc^x$	–	87.563	1.139	0.283	69.578%	73.011	40.963%	42.640
$y = bx^c$	–	84.896	0.517	0.378	65.260%	66.014	36.037%	34.119
$y = a + bx^c$	377.337	-279.211	-0.141	0.422	66.426%	62.681	33.940%	29.944

THE AEROSPACE CORPORATION

DERIVING PPT # 49

MUPE (IRLS) Regression Functions Fit To Example 3 Data

(Minimum Unbiased Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	52.390	–	0.317	93.878%	78.144	0.000%	26.444
$y = a + bx$	12.634	34.730	–	0.317	88.529%	59.483	0.000%	6.054
$y = a + b \log x$	62.964	65.582	–	0.420	78.072%	51.538	0.000%	-3.989
$y = bc^x$	–	30.254	1.468	0.216	95.767%	63.431	0.000%	4.155
$y = bx^c$	–	52.303	0.674	0.359	83.208%	53.789	0.000%	1.489
$y = a + bx^c$	-1.145	53.734	0.658	0.361	87.059%	56.240	0.000%	1.384

THE AEROSPACE CORPORATION

DERIVING PPT # 50

ZPB/MPE Regression Functions Fit To Example 3 Data

(Zero Percentage Bias, Minimum Percentage Error)

FUNCTION	a	b	c	R^2	PCT SEE	ADD SEE	PCT BIAS	ADD BIAS
$y = bx$	–	52.390	–	0.317	93.878%	78.145	0.000%	26.444
$y = a + bx$	25.018	23.747	–	0.317	87.527%	54.659	0.000%	-2.100
$y = a + b \log x$	62.716	64.137	–	0.420	78.034%	51.723	0.000%	-4.337
$y = bc^x$	–	51.695	1.139	0.283	90.554%	59.176	0.000%	-4.126
$y = bx^c$	–	54.302	0.517	0.378	81.599%	52.302	0.000%	-3.952
$y = a + bx^c$	-67.649	126.666	0.247	0.405	82.786%	53.694	0.000%	-3.087

DERIVING PPT # 51

THE AEROSPACE CORPORATION